BECOMING MORE POSITIVE WITH NEGATIVES

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This paper is based on a recently started Doctor of Education research project at the University of Melbourne. The project is being supervised by Professor Kaye Stacey.

Abstract

The research project discussed in this paper re-opens the issue of teaching negative number concepts and operations. The experimental work in schools has only recently commenced. In conjunction with a detailed evaluation of current teaching strategies, an experimental card/counter teaching strategy will be carefully 'classroom tested'. Short and long term student learning outcomes resulting from the experimental and the more common current teaching methods will be compared and contrasted. It is anticipated that the study will provide enhanced insights relating to the teaching and learning of negative numbers resulting in substantially improved teaching practice and student learning.

Introduction

Teaching mathematics to trainee teachers, engaged in Vocational Education studies and the teaching of 'trade mathematics' to apprentices and other TAFE level student groups, has indicated a widespread lack of understanding of number concepts and operations, particularly with regard to negative numbers. Typically, trainee teachers themselves seem to lack in-depth understanding although they may be able to quote and use the 'rules'. The knowledge levels of their students (many of whom claim to have recently 'studied' mathematics up to at least year eleven) is, at best, similar but often lower. The students often admit to anxiety about this 'problem'. Sheila Tobias (1982) in her book *Overcoming Math Anxiety* draws particular attention to the difficulties caused by the 'The Many Meanings of Minus' (p. 172). Some adult learners have claimed that this was one of the factors that turned them 'off maths'. Although, to the mathematically capable, it may be perhaps a rather small and even trivial issue ("I always remember the rules and get it right so where is the problem?"), the number of questions received about the topic, from puzzled students, over an extended period of time, provides convincing evidence that it has a basic level of importance that should not be ignored.

Common difficulties

In particular, areas of difficulty, confusion and lack of understanding that appear to have developed from 'traditional' teaching strategies for such former secondary level students include:

* subtraction of negative numbers ("I think it becomes positive but I don't know (or can't understand) why!"),
* addition of pairs of negative numbers ("Its positive because two minuses make a plus!"),
* multiplication of pairs of negative numbers (similar response to the above),

* misunderstandings that seem to result from the dual use of the 'minus' sign (binary usage - for the purpose of denoting subtraction and unary usage - for the 'labelling' of negative numbers).
The loose use of language in this context by both teachers and students may also hinder understanding. Translating '6 - ⁻3', as "six minus minus three" instead of "six minus negative three" is an example.

Common mistakes include mis-application of perhaps rotely taught and vaguely remembered and interpreted 'rules' such as:

"like signs positive"
"unlike signs negative".

On the assumption that a full and thorough understanding of number concepts and operations is a fundamental requirement and an essential step in the development of mathematical competency and, according to Paulos (1990), the avoidance of mathematical illiteracy, it appears that there is a need to investigate current teaching practices, identify weaknesses and shortcomings and to develop more effective teaching and learning strategies for the topic.

Difficulties with the 'idea' of negative number and the teaching of negative number concepts are not new. There exists, in fact, a considerable amount of literature on the discussion of such difficulties, both historical (eg. Maclaurin (1748) in *A Treatise of Algebra in Three Parts* and Euler (1767) in his *Vollstandige Anleitung zur Algebra*, both cited and discussed in Katz, 1993) and relatively recent. An extensive array of teaching strategies concerned particularly with tackling the 'hard' areas identified above have been tried. For a review of a sample of such literature see Hayes, 1992a. In spite of this a general lack of understanding of the topic still prevails. Perhaps most of the current teaching strategies used and appearing in textbooks are not sufficiently basic and a more fundamental and practical approach is required.

The Lack of Recent Research

Very little in the way of serious and rigorous research appears to have been done on the teaching of negative numbers within the last decade. Bell (1983) reviewed the topic and provided a critical discussion of some of the teaching strategies used. Bell draws attention to, and also appears to generally support the French theorists' (eg. Glaesser, 1981) view, that no single integer 'model' should be sought or can be used to cover all operations. The serious obstacle is that of multiplication of pairs of negative numbers. With regard to the latter 'problem' an extensive array of teaching strategies have been devised. Scopes (1973) outlines several and Freudenthal (1983) provides a useful theoretical review. Arcavi and Bruckheimer (1981) give a useful categorisation scheme in a report dealing with their attempt to research and compare the relative effectiveness of some of the teaching methods. However their research provided little in the way of evidence for either a 'best' teaching strategy or much to suggest which strategies do tend to provide better learning outcomes.

The CSMS project team at Chelsea College, University of London, included testing of pupils' understanding of negative numbers (Kuchemann, 1981). In discussing the results Kuchemann makes the following recommendation concerning the abandonment of the popular number line teaching strategy:

> *For addition the model is extremely straightforward and effective..... However, for subtraction the model is far more difficult to use, not only because the operation is not seen as a simple sequence but also because the meanings given to the integers differ and are not consistent with the simple meaning used for addition.*
>
> *This change in meaning suggests that the number line should be abandoned despite its proven effectiveness for addition, in favour of a more consistent approach, for example one in which the integers are regarded as discrete entities or objects, constructed in such a way that the positive integers cancel out the negative integers. The clear advantage of such a model is that the same meaning can be used for the integers both within and across the operations of addition and subtraction, and it seems likely that this would enhance children's understanding of subtraction in particular.* (Kuchemann in Hart (ed.), 1981, p. 87)

Kuchemann then warns that the hopes placed in such a replacement strategy should not be exaggerated and implies the need for **careful investigation**. He then refers to the 'problem' of finding a suitable model for multiplication concluding with the comment:

> *....it is difficult to see a genuine way round this limitation without using an entirely abstract approach which, it has already been argued, is likely to leave most children with no understanding of integers at all.*

Implications

Freudenthal (1973) supported by Fischbein (1987) argued for the use of a 'logico-deductive' teaching strategy, particularly for developing the rule for the multiplication of pairs of negative integers. It is my belief that the card/counter model of integers is an embodiment that can be used to enlighten the basic number laws and that, provided that pupils thoroughly understand the laws fully for positive number operations, the model can be used to meaningfully facilitate the 'logical deduction' of the negative number multiplication rules.

A teaching and learning model based on counting

Most children find mathematics to be meaningful and understandable only when it is seen as applicable to or embodied in 'objects'. Fundamental concepts (eg. numbers, points, lines, angles, length, area, volume) and operations (eg. addition, subtraction, multiplication, division) of mathematics require 'practical' application and modelling for the development of purpose and understanding. Support for this view is contained in the writings of theorists such as Davis, Maher and Noddings (1990), Dienes (1959, 1971), Fischbein (1987), Freudenthal (1973, 1983, 1991), Gattegno (1960), Lovell (1971), Piaget (1952) and Skemp (1986).

The fundamental activity basic to the understanding of number concepts, properties and operations is counting. Richard Skemp makes the claim that;

> *Most people, if asked, what are the ideas with which mathematics begins, would reply 'numbers' or possibly 'counting'. We shall therefore begin our conceptual analysis with these two closely connected ideas. Before beginning, however, it is worth warning the reader that the ideas which will be introduced are elementary in the sense of basic, but not in the sense of easy. It is sometimes harder to explain something simple (how does a wheel work?) than something more complex.* (Skemp, 1986, p. 133)

Concerning the relationship between number and counting, Skemp continues on the same page with the following assertion;

> *Number and counting are by no means inseparable. It is possible to have a rudimentary idea of a number without being able to count and Piaget has shown that children can count in a restricted sense without really having the concept of number. But if by counting we mean something like 'finding the number of apples in a bowl', then it is clear that counting in its everyday meaning is a way of finding a certain property of a collection of objects, which we call number. This implies that number and counting are ideas which belong closely together, and that, of the two, number is the more basic.*

Practically all cases and situations which involve the 'basic' operations with natural numbers (positive integers) could be evaluated by the process of counting. For example, **addition** can be regarded as combining specified sized 'collections' and counting the contents of the 'new' larger collection; **subtraction** as the removal of a specified 'quantity of material' from a 'large' collection and counting the remainder; **multiplication** as the combining of equal sized piles and counting the total in the single large pile (a process of repeated addition); **division** may be interpreted as counting how many times (without replacement) a specified sized smaller heap can be taken away from a larger heap (repeated subtraction).

It is contended that using the above interpretations in conjunction with cards or counters (or perhaps other manipulable materials eg. jelly beans, bottle tops, icy pole sticks) can be used to facilitate understanding of negative number concepts and operations.

It appears that the basic natural number concepts develop (or are expected to develop) out of counting activities in the early primary school years. Early number concept development and fluency is usually assisted by primary class counting both forward and backward. Backward counting activities may be extended, initially to zero, and later to negatives. Pupils may become aware of negative number applications such as temperature scales and number 'ordering' activities. (Are children living in sub-zero regions mathematically advantaged?) A few years later, usually in early secondary school years, the concept of negative numbers is more formally introduced.

The Project

In reopening issues relating to the teaching of negative numbers the 'card/counter' teaching strategy will be compared the more common teaching methods currently used. There are two major aims:

a) To make a detailed, school-based study of teaching strategies currently being used for the teaching of negative number concepts and operations and to investigate pupil depth of understanding resulting from the current strategies. (Which of the present 'popular' teaching methods generally seem to 'work' best?)

b) To experimentally investigate and develop a 'card/counter' method of teaching negative number concepts and operations and thus compare the relative effectiveness of this strategy with the other teaching methods.

The Cards/Counter Teaching Strategy

The cards/counter method of teaching integers uses either cards (labelled '$^{+}1$', '$^{-}1$', '0') or coloured counters (eg. black, red, white) and provides an easily manipulable 'model' of integers using counting as its basic initial 'hands-on' operational activity. Its potential for producing better understanding of number concepts and operations has not been thoroughly explored and researched. It may provide, if used a carefully sequenced manner, an initial teaching strategy that may be more simple and more meaningful than other teaching strategies commonly used.

Previous use of the method

The method has been used with trainee teachers, taking the elective Mathematics in Vocational Education as a unit in the Diploma of Teaching (TAFE), at the Hawthorn Institute of Education and also with Graduate Diploma in Mathematics Education students. A report of a small pilot study done with such students is given in Hayes, 1992b. The strategy was used at Swinburne Technical School several years ago with a school-based mathematics method group of Grad. Dip. Ed. students working with a year eight class. An article on a related (jelly-beans) interpretation of the strategy recently appeared in *The Australian Mathematics Teacher* (Hollingsworth, 1992). Alan Bell (1983) mentions some experiments, using coloured cubes, done by Tim Rowland of Homerton College, Cambridge, involving primary teachers and pupils. In referring to the 'method' Bell makes the following remark.

> *A type of embodiment of directed number which has been mentioned from time to time but not, I think, much developed is the 'coloured cubes' model, one colour being taken as positive and the other negative.* (Bell, 1983, p.32)

Freudenthal (1983), discussing old and new models for teaching, in a chapter entitled 'Negative Numbers and Directed Magnitudes' in his book *Didactical Phenomenology of Mathematical Structures*, makes the following comments.

The fault of the (old) models, dealt with so far, is the didactical asymmetry between positive and negative numbers. The positive numbers are more concrete in the sense of greater originality; so one can operate with them; the negative numbers are secondary, introduced as results of operations, which formerly were impossible, fit to be operated on if need be, but unfit to have operations performed on them. In other words, the positive numbers are active, the negative numbers only passive.

If rather than asking for a model, one is satisfied with the formalism of what I call the algebraic permanence principle, this difference is non-existent as soon as one has decided about extending, the negative numbers have the same legal status as the positive ones; operating with negative numbers is formally justified and in no way distinguished from that with positive ones. (Freudenthal, 1983, p. 438)

Continuing, on the same page, under the heading of 'New Models', Freudenthal makes the following key assertion.

*If rather than being satisfied with the algebraic permanence principle, one looks for more satisfying models (*my emphasis*) than those dealt with so far, it is now clear that positive and negative numbers shall be given the same opportunity. The former models admitted a symmetry between adding and subtracting as inverses of each other - the one undoes the other. What is asked for is an equal status for positive and negative numbers, such that the operations can be performed by means of this equal status.*

Freudenthal then suggests, as a non-geometric example, a method of teaching using positive and negative (black and red) counters (which he attributes to Gattegno); substantially the method that I am proposing to develop and investigate as the major focus of this study. He makes the point that this embodiment means that integers are considered as ordered pairs of natural numbers with the equivalence relation

$$[a,b] \sim [c,d] \leftrightarrow a + d = b + c$$

being an equivalence class, or at least operating in a way that, mathematically conceptualised, is known as equivalence class formation.

The model provided is a black and red counter strategy (positive and negative counters) based on the following 'rules';

a black and red one can annihilate each other,

and conversely

a black-red pair can come into being from nothing.

This approach, using both counters and cards as the manipulable materials, is the basis of the teaching and learning strategy that is being investigated. I am suggesting the use of the card/counter approach as the initial operational secondary level teaching strategy. The degree to which the approach should then be supplemented and reinforced by other embodiments and negative number applications will be carefully considered and also investigated. Dienes (1971) recommends the use of 'multiple embodiments' and experiential structured materials to facilitate depth of understanding in the teaching of most topics. In particular, a vector approach he outlines for the modelling of integer concepts and operations seems closely related to the cards/counter strategy and could be a useful extension. Further embodiments, applications, interpretations and illustrations (eg. use of the number line, kinematic examples, graphs and use of the cartesian plane, reflective images, extending number patterns, credits and debits, use of calculators etc.) which I consider to be more advanced and supplementary in nature could then follow.

Learning Indicators

Indicators of student knowledge and understanding that will be used to answer key research questions and prove or disprove the above hypotheses may include the ability to;

* appropriately use, in both the short and long term, the card/counter model to demonstrate and orally explain negative number and related concepts (eg. additive inverse and zero, equivalent valued 'collections') and operations (eg. "Use the cards to demonstrate the value of '3 - $^-5$'.")

* translate card/counter activities into correct (and correctly sequenced) written 'mathematical statements',

* develop (discover) as a 'natural' and meaningful consequence of the card/counter activities the correct 'rules' for negative number operations and to be able to correctly apply the 'rules' to written calculations,

* show evidence of being able to reason and construct logically sequenced and correct mathematical statements involving negative number calculations without using the cards or counters (eg. "Write a sequence of mathematical sentences to show why 0 - $^-7$ = 7."),

* successfully transfer the concepts and operations learned to other negative number embodiments (eg. number line) and applications (eg. 'banking', manipulating and simplifying mathematical expressions, solving equations, graphing and the cartesian plane etc.),

* recognise and make confident and appropriate use of zero to perform negative number 'calculations' initially with, and later without, using the cards or counters,

* consistently make correct responses to appropriate oral and written test items.

Concluding remarks

The field work has only recently commenced. It is expected to continue for the next two years in at least three schools with about two hundred students involved in the cards/counter component of the study. The overall conjecture is that, following appropriate use of the card/counter teaching and learning approach, students will be much less likely to make common errors and show misunderstandings that seem to result from other current negative number teaching strategies (eg. adding negative integers to get a positive answer). It is also contended that such deeper understanding will transfer to algebraic manipulative skills such as the expansion of brackets when minus signs are involved. Ideally, of course, the intention is to provide all students with a better opportunity to be able to know, understand and correctly and automatically apply 'the rules'. Freudenthal (1983) refers to this learning as the development of 'automisms'.

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